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Maximal sector of analyticity for C_0-semigroups generated by elliptic operators with separation property in L^p

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Abstract. Analytic continuation of the C_0-semigroup $\{e^{-zA}\}$ on $L^p(\mathbb{R}^N)$ generated by the second order elliptic operator $-A$ is investigated, where A is formally defined by the differential expression $Au = -\mathrm{div}(a\nabla u) + (F\cdot\nabla)u + Vu$ and the lower order coefficients have singularities at infinity or at the origin.

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1 Introduction

In this paper we deal with general second order elliptic operators of the form

$$(Au)(x) := -\mathrm{div}(a(x)\nabla u(x)) + (F(x)\cdot\nabla)u(x) + V(x)u(x), \quad x \in \mathbb{R}^N,$$

where $N \in \mathbb{N}$, $a \in C^1 \cap W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^{N\times N})$, $F \in C^1(\Omega; \mathbb{R}^N)$ and $V \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R})$ and the choice of $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N \setminus \{0\}$ depends on the location of the

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singularities of F and V. Under the assumption on the triplet (a, F, V) specified below we discuss the maximal sector of analyticity for the semigroup $\{T_p(t)\}$ on $L^p = L^p(\mathbb{R}^N)$ $(1 < p < \infty)$ generated by $-A$ with a suitable domain. Because the domain of A changes with the choice of Ω, we describe it when we state the respective result.

The purpose of this paper is to improve the known sector of analyticity for $\{T_p(t)\}$. In Metafune-Pallara-Prüss-Schnaubelt [10] and Metafune-Prüss-Rhandi-Schnaubelt [11], it is proved that $\{T_p(t)\}$ is analytic and contractive in $\Sigma(\eta_p)$, where

$$\begin{aligned}\Sigma(\eta) &:= \{z \in \mathbb{C} \setminus \{0\}\ ;\ |\arg z| < \eta\},\\ \eta_p &:= \frac{\pi}{2} - \tan^{-1}\sqrt{\frac{(p-2)^2}{4(p-1)} + \frac{\beta^2}{4(1-\theta/p)}}\end{aligned}$$

for some $\beta \geq 0$ (see (2.1) below) and $\theta < p$ (satisfying $\theta V \geq \operatorname{div} F$); note that η_p is smaller than

$$\omega_p := \frac{\pi}{2} - \tan^{-1}\Big(\frac{|p-2|}{2\sqrt{p-1}}\Big)$$

which is the angle of contractivity for C_0-semigroups generated by Schrödinger operators (see, e.g., Okazawa [12]). Using Gaussian estimates, one can construct a non-contractive holomorphic extension of $\{T_p(t)\}$ to $\Sigma(\eta)$ with $\eta \geq \eta_p$, where η is independent of p. However, an application of results in Ouhabaz [13, 14] would give $\eta = \eta_2$. We instead prove $\eta = \eta_{\bar{p}}$ for a certain $\bar{p}$ and show that $\bar{p}$ can be different from 2, see Remark 3 below.

2 Description of our assumption

Let $A_{p,\max}$ and A_p be the operators respectively defined as follows:

$$\begin{aligned} A_{p,\max}u &:= Au, \quad D(A_{p,\max}) := \{u \in L^p \cap W^{2,p}_{\mathrm{loc}}(\Omega);\ Au \in L^p\},\\ A_p u &:= Au, \quad D(A_p) := W^{2,p}(\mathbb{R}^N) \cap D(F\cdot\nabla) \cap D(V),\end{aligned}$$

where $D(F\cdot\nabla) := \{u \in L^p \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N); (F\cdot\nabla)u \in L^p\}$ and $D(V) := \{u \in L^p; Vu \in L^p\}$.

Now we present the basic assumption on the triplet (a, F, V) defining $A_{p,\max}$ and A_p. As in Introduction Ω stands for $\mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$.

(H1) ${}^t a = a \in C^1 \cap W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^{N\times N})$ and a is uniformly elliptic on $\mathbb{R}^N$, that is, there exists a constant $\nu > 0$ such that

$$\langle a(x)\xi, \xi\rangle \geq \nu|\xi|^2, \quad x \in \mathbb{R}^N, \quad \xi \in \mathbb{C}^N,$$

where $\langle\cdot,\cdot\rangle$ is the usual Hermitian product;

(**H2**) $F \in C^1(\Omega;\mathbb{R}^N)$, $V \in L^\infty_{\text{loc}}(\Omega;\mathbb{R})$ and there exist three constants $\beta \geq 0$, γ_1, $\gamma_\infty > 0$ and a **nonnegative** auxiliary function $U \in L^\infty_{\text{loc}}(\Omega)$ such that

$$|\langle F(x),\xi\rangle| \leq \beta U(x)^{\frac{1}{2}}\langle a(x)\xi,\xi\rangle^{\frac{1}{2}} \quad \text{a.a. } x\in\Omega,\ \xi\in\mathbb{C}^N, \tag{2.1}$$

$$V(x) - \operatorname{div}F(x) \geq \gamma_1 U(x) \quad \text{a.a. } x\in\Omega, \tag{2.2}$$

$$V(x) \geq \gamma_\infty U(x) \quad \text{a.a. } x\in\Omega; \tag{2.3}$$

(**H3**) the auxiliary function $U \geq 0$ in (**H2**) belongs to $C^1(\Omega;\mathbb{R})$ and there exist constants $c_0 \geq k_0 := \max\{\gamma_1,\gamma_\infty\} > 0$ and $c_1 \geq 0$ such that

$$V(x) \leq c_0 U(x) + c_1 \quad \text{a.a. } x\in\Omega \tag{2.4}$$

and U satisfies an **oscillation condition** with respect to the diffusion a, that is,

$$\lambda_0 := \lim_{c\to\infty}\left(\sup_{x\in\Omega}\frac{\langle a(x)\nabla U(x),\nabla U(x)\rangle^{1/2}}{(U(x)+c)^{3/2}}\right) < \infty. \tag{2.5}$$

This yields a working form of the oscillation condition: for every $\lambda > \lambda_0$ there exists a constant $C_\lambda > 0$ such that

$$\langle a(x)\nabla U(x),\nabla U(x)\rangle^{1/2} \leq \lambda(U(x)+C_\lambda)^{3/2}, \quad x\in\Omega. \tag{2.6}$$

In particular, if $\Omega = \mathbb{R}^N\setminus\{0\}$ then $U(x)$ is assumed to tend to infinity as $x\to 0$.

Example 1 (Maeda-Okazawa [9])**.** Put $a_{jk} = \delta_{jk}$. Then it is possible to compute λ_0 for $U(x) := |x|^\alpha$ when $\alpha \notin (-2,1]$.

(**i**) Let $U(x) := |x|^\alpha$ ($\alpha > 1$). Then $U \in C^1(\mathbb{R}^N)$ and $\lambda_0 = 0$. In fact, we have

$$\frac{\langle a(x)\nabla U(x),\nabla U(x)\rangle^{1/2}}{(U(x)+c)^{3/2}} = \frac{\alpha|x|^{\alpha-1}}{(|x|^\alpha+c)^{3/2}} \leq \alpha c^{-1/2-1/\alpha} \to 0 \ (c\to\infty).$$

(**ii**) Let $U(x) := |x|^{-\beta}$ ($\beta > 2$). Then $U \in C^1(\mathbb{R}^N\setminus\{0\})$ and $\lambda_0 = 0$. The computation is similar as above. In particular, if $\beta = 2$, then $\lambda_0 = 2$.

Remark 1. Let $\lambda > \lambda_0$ and $C_\lambda > 0$ as in (2.6) and put

$$\tilde{U}(x) := U(x) + C_\lambda > 0 \quad \text{on } \Omega.$$

Then $\tilde{U}$ plays the role of a **positive** auxiliary function for the new (formal) operator

$$\tilde{A} := A + k_0 C_\lambda$$

with modified potential

$$\tilde{V}(x) := V(x) + k_0 C_\lambda > 0 \quad \text{on } \Omega,$$

where k_0 is as in condition (**H3**). In fact, the new triplet $(a, F, \tilde{V})$ satisfies the original inequalities (2.1)–(2.4) with the pair (U, V) replaced with $(\tilde{U}, \tilde{V})$:

$$|\langle F(x), \xi\rangle| \le \beta (U(x) + C_\lambda)^{\frac{1}{2}} \langle a(x)\xi, \xi\rangle^{\frac{1}{2}}, \tag{2.1$'$}$$

$$[V(x) + k_0 C_\lambda] - \operatorname{div} F(x) \ge \gamma_1 (U(x) + C_\lambda), \tag{2.2$'$}$$

$$V(x) + k_0 C_\lambda \ge \gamma_\infty (U(x) + C_\lambda), \tag{2.3$'$}$$

$$V(x) + k_0 C_\lambda \le c_0 (U(x) + C_\lambda) + c_1. \tag{2.4$'$}$$

Note further that (2.6) is also written in terms of $\tilde{U}$:

$$\langle a(x)\nabla \tilde{U}(x), \nabla \tilde{U}(x)\rangle^{1/2} \le \lambda\, \tilde{U}(x)^{3/2} \quad \text{on } \Omega. \tag{2.6$'$}$$

In particular, (2.1$'$) and (2.6$'$) yield that

$$|(F \cdot \nabla)\tilde{U}(x)| \le \beta\lambda \tilde{U}(x)^2 \quad \text{on } \Omega. \tag{2.7}$$

3 The operators with singularities at infinity

In this section we consider the case where $\Omega = \mathbb{R}^N$.

Theorem 1. *Assume that conditions* (**H1**) *and* (**H2**) *are satisfied with* $\Omega = \mathbb{R}^N$. *Then one has the following assertions*:

(**i**) *Let* $1 < q < \infty$. *Then* $A_{q,\max}$ *is* m*-sectorial in* L^q, *that is,* $\{e^{-zA_{q,\max}}\}$ *is an analytic contraction semigroup on* L^q *on the closed sector* $\overline{\Sigma}(\pi/2 - \tan^{-1} c_{q,\beta,\gamma})$, *where*

$$c_{q,\beta,\gamma} := \sqrt{\frac{(q-2)^2}{4(q-1)} + \frac{\beta^2}{4}\left(\frac{\gamma_1}{q} + \frac{\gamma_\infty}{q'}\right)^{-1}} \tag{3.1}$$

and q' *is the Hölder conjugate of* q. *Moreover,* $C_0^\infty(\mathbb{R}^N)$ *is a core for* $A_{q,\max}$.

(**ii**) *Let* $p \in (1, \infty)$ *be arbitrarily fixed. Then the semigroup* $\{e^{-zA_{p,\max}}\}$ *in assertion* (**i**) *admits an analytic continuation to the open sector* $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$, *where*

$$K_{\beta,\gamma} := \min_{1<q<\infty} c_{q,\beta,\gamma}. \tag{3.2}$$

Moreover, there exists a constant $\omega_0 > 0$ *such that* $\{e^{-z(\omega_0 + A_{p,\max})}\}$ *forms a bounded analytic semigroup on* L^p :

$$\|e^{-zA_{p,\max}}\|_{L^p} \le M_\varepsilon e^{\omega_0 \operatorname{Re} z} \quad \text{on } \overline{\Sigma}(\pi/2 - \tan^{-1} K_{\beta,\gamma} - \varepsilon). \tag{3.3}$$

Here the constant ω_0 *depends only on* N, $\|a_{jk}\|_{L^\infty(\mathbb{R}^N)}$ *and* $\|\nabla a_{jk}\|_{L^\infty(\mathbb{R}^N)}$, *while the constant* $M_\varepsilon \geq 1$ *depends only on* ε, N, ν, β, γ_1, γ_∞ *and* $\|a_{jk}\|_{L^\infty(\mathbb{R}^N)}$.

(**iii**) *Assume further that* (**H3**) *is satisfied with* $\Omega = \mathbb{R}^N$. *If*

$$(p-1)\lambda_0\left(\frac{\beta}{p} + \frac{\lambda_0}{4}\right) < \frac{\gamma_1}{p} + \frac{\gamma_\infty}{p'}, \tag{3.4}$$

then $A_{p,\max}$ *has the so-called separation property*:

$$\|\mathrm{div}(a\nabla u)\|_{L^p} + \|(F\cdot\nabla)u\|_{L^p} + \|Vu\|_{L^p} \leq C\|(1+A_{p,\max})u\|_{L^p} \tag{3.5}$$

for all $u \in D(A_{p,\max})$ *which implies the coincidence* $A_{p,\max} = A_p$ *and hence* $\{e^{-zA_p}\}$ *is analytic in* $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$.

Here three remarks are in order.

Remark 2. Assertion (**i**) is a particular case of [15, Theorem 1.3]; note that the sector of analyticity and *contraction property* for $\{e^{-zA_{p,\max}}\}$ is reduced to the positive real axis (that is, $\tan^{-1} c_{p,\beta,\gamma} \to \pi/2$) as p tends to 1 or to ∞.

Remark 3. Assertion (**ii**) states that $\{e^{-zA_{p,\max}}\}$ admits an analytic continuation without *contraction property* (in general) to a p-independent sector $\Sigma(\pi/2-\tan^{-1} K_{\beta,\gamma})$ bigger than $\Sigma(\pi/2-\tan^{-1} c_{q,\beta,\gamma})$. Moreover, in general the constant $c_{2,\beta,\gamma}$ does not attain $\min_{1<q<\infty} c_{q,\beta,\gamma}$ ($= K_{\beta,\gamma}$). In fact, we see by a simple calculation that

$$\frac{\partial (c_{q,\beta,\gamma})^2}{\partial q} = \frac{q(q-2)}{4(q-1)^2} + \frac{\beta^2(\gamma_1-\gamma_\infty)}{4q^2}\left(\frac{\gamma_1}{q} + \frac{\gamma_\infty}{q'}\right)^{-2}.$$

Therefore if $\gamma_1 \neq \gamma_\infty$, then we have

$$\frac{\partial (c_{q,\beta,\gamma})^2}{\partial q}\Big|_{q=2} = \frac{\beta^2(\gamma_1-\gamma_\infty)}{4(\gamma_1+\gamma_\infty)^2} \neq 0.$$

This implies that in the case where $\gamma_1 \neq \gamma_\infty$ the sector derived by L^p-theory can be bigger than the one derived by L^2-theory. Consequently, we have $c_{2,\beta,\gamma} > K_{\beta,\gamma}$. An example with $\gamma_1 \neq \gamma_\infty$ is also given later (see Example 3 below in Section 4).

Remark 4. It is shown in [10] that A_p is m-sectorial of type $S(\tan\omega)$ in L^p, where

$$\omega := \tan^{-1} c_{p,\beta,\gamma} > \omega_p = \tan^{-1}\frac{|p-2|}{2\sqrt{p-1}},$$

if p satisfies (3.4). Their proof is based on a perturbation technique with the separation property (3.5) under a setting similar to assertion (**iii**). Theorem 1 makes it clear that (3.5) is necessary only for the domain characterization of A_p.

First we describe the key lemma as Lemma 1 which plays an essential role in proving the existence of analytic continuation for $\{e^{-zA_{p,\max}}\}$. Lemma 1 transplants a bounded analytic semigroup on L^{p_0} onto L^p without changing the sector (or angle) of analyticity. Note that Lemma 1 was first proved in Ouhabaz [13] (for $A_{2,\max}$ associated with symmetric forms), and then in Arendt-ter Elst [2] and Hieber [8].

Lemma 1. *For some* $p_0 \in (1,\infty)$ *let* $\{T_{p_0}(t);\, t \geq 0\}$ *be a* C_0*-semigroup on* L^{p_0}*.*
(**i**) (*Gaussian Estimate*) *Assume that* $\{T_{p_0}(t)\}$ *admits a Gaussian estimate with integral kernel* $\{k_t\}$*. For every* $p \in (1,\infty)$ *define the family* $\{T_p(t); t \geq 0\}$ *as* $T_p(0)f := f$ *and*

$$\big(T_p(t)f\big)(x) := \int_{\mathbb{R}^N} k_t(x,y)f(y)\,dy \quad \text{a.a. } x \in \mathbb{R}^N, \quad f \in L^p, \ \ t > 0.$$

Then the new family $\{T_p(t)\}$ *forms a* C_0*-semigroup on* L^p*.*

(**ii**) (*Analyticity*) *Assume further that* $\{e^{-\omega_0 z}T_{p_0}(z)\}$ *is a bounded analytic semigroup on* L^{p_0} *in the sector* $\Sigma(\psi_0)$ *such that for every* $\varepsilon > 0$ *there exists a constant* $M_\varepsilon \geq 1$ *satisfying*

$$\|T_{p_0}(z)\|_{L^{p_0}} \leq M_\varepsilon e^{\omega_0 \mathrm{Re}\, z} \quad \forall\, z \in \overline{\Sigma}(\psi_0 - \varepsilon). \tag{3.6}$$

Then $\{T_p(t)\}$ *has almost the same property as* $\{T_{p_0}(t)\}$*; namely,* $\{e^{-\omega_0 t}T_p(t)\}$ *can be extended to a bounded analytic semigroup* $\{e^{-\omega_0 z}T_p(z)\}$ *in the sector* $\Sigma(\psi_0)$ *such that for every* $\varepsilon > 0$ *there exists* $\tilde{M}_\varepsilon \geq 1$ *satisfying*

$$\|T_p(z)\|_{L^p} \leq \tilde{M}_\varepsilon e^{\omega_0 \mathrm{Re}\, z} \quad \forall\, z \in \overline{\Sigma}(\psi_0 - \varepsilon)$$

(*which is nothing but* (3.6) *with* p_0 *and* M_ε *replaced with* p *and* $\tilde{M}_\varepsilon$*, respectively*)*, where the constant* $\tilde{M}_\varepsilon$ *depends only on* ε*,* N*,* p_0*,* ψ_0*,* M_ε*,* C *and* b*.*

Next we note that the (analytic contraction) semigroup $\{e^{-tA_{2,\max}}\}$ admits a Gaussian estimate. The proof of the following lemma is given in [3, Theorem 4.2].

Lemma 2. *Assume that* (**H1**)*,* (**H2**) *and* (**H3**) *are satisfied with* $\Omega = \mathbb{R}^N$*. Then* $\{e^{-tA_{2,\max}}\}$ *admits a Gaussian estimate with* **nonnegative** *kernel* $\{k_t\}$ *satisfying*

$$0 \leq k_t(x,y) \leq Ct^{-N/2}\exp\Big(\omega_0 t - \frac{|x-y|^2}{bt}\Big) \quad \text{a.a. } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where the constant ω_0 *depends only on* N*,* $\|a_{jk}\|_{L^\infty}$ *and* $\|\nabla a_{jk}\|_{L^\infty}$*, while* C*,* b *depend only on* N*,* ν*,* β*,* γ_1*,* γ_∞ *and* $\|a_{jk}\|_{L^\infty}$*.*

Next we state a modification of [10, Lemma 2.3]; note that the constant factors in the inequalities are figured out. It is worth noticing that under conditions (**i**) and (**ii**)

$$A_{p,\min} := A, \quad D(A_{p,\min}) := C_0^\infty(\mathbb{R}^N),$$

is accretive in L^p (see, e.g., [10, Proposition 2.2] or [15, Theorem 1.1]).

Lemma 3. *Assume that* (**H1**), (**H2**) *and* (**H3**) *are satisfied with* $\Omega = \mathbb{R}^N$. *Put*

$$k_p(\lambda) := \Big(\frac{\gamma_1}{p} + \frac{\gamma_\infty}{p'}\Big) - (p-1)\lambda\Big(\frac{\beta}{p} + \frac{\lambda}{4}\Big), \quad \lambda > \lambda_0,$$

and let C_λ *be a constant in* (2.6). *If* $k_p(\lambda) > 0$, *then for every* $\xi > k_0 C_\lambda$ $(= C_\lambda \max\{\gamma_1, \gamma_\infty\})$ *and* $u \in C_0^\infty(\mathbb{R}^N)$ *one has*

$$\|(U + C_\lambda)u\|_{L^p} \le \frac{1}{k_p(\lambda)} \|(\xi + A)u\|_{L^p}, \tag{3.7}$$

$$\|(F\cdot\nabla)u\|_{L^p} + \|(V + k_0 C_\lambda)u\|_{L^p}$$
$$\le 2\left(1 + \frac{c_0 + \beta\,\tilde C_{1/(2\beta)}}{k_p(\lambda)} + \frac{c_1}{\xi - k_0 C_\lambda}\right)\|(\xi + A)u\|_{L^p}, \tag{3.8}$$

where $\tilde C_{1/(2\beta)} > 0$ *depends only on* N, p, ν *and* $\|a_{jk}\|_{W^{1,\infty}}$. *Moreover, let* $\xi \ge 1 + k_0 C_\lambda$. *Then there exists* $C > 0$ *such that for every* $u \in C_0^\infty(\mathbb{R}^N)$,

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} \le C\Big(5 + 2\frac{c_0 + \beta\,\tilde C_{1/(2\beta)}}{k_p(\lambda)} + \frac{2c_1}{\xi - k_0 C_\lambda}\Big)\|(\xi + A)u\|_{L^p}, \tag{3.9}$$

where $C > 0$ *depends only on* N, p, ν *and* $\|a_{jk}\|_{W^{1,\infty}}$.

Proof. Define $\tilde A u := (A + k_0 C_\lambda)u$ for $u \in C_0^\infty(\mathbb{R}^N)$ and set $\eta := \xi - k_0 C_\lambda > 0$. Then $(\eta + \tilde A)u = (\xi + A)u$ so that (3.7) and (3.8) are respectively equivalent to

$$\|\tilde U u\|_{L^p} \le k_p(\lambda)^{-1}\|(\eta + \tilde A)u\|_{L^p}, \tag{3.10}$$

$$\|(F\cdot\nabla)u\|_{L^p} + \|\tilde V u\|_{L^p}$$
$$\le 2\big(1 + k_p(\lambda)^{-1}[c_0 + \beta\,\tilde C_{1/(2\beta)}] + \eta^{-1}c_1\big)\|(\eta + \tilde A)u\|_{L^p}, \tag{3.11}$$

where $\tilde U = U + C_\lambda > 0$ and $\tilde V = V + k_0 C_\lambda > 0$ (see Remark 1).

First we prove (3.10). We use the key identity in [15, Section 1]: for every $u \in C_0^\infty(\mathbb{R}^N)$, $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and $1 \le r \le \infty$,

$$\int_{\mathbb{R}^N} (Au)\overline{v}\,dx = \int_{\mathbb{R}^N}\left[\langle a\nabla u, \nabla v\rangle + \left(V - \frac{\operatorname{div} F}{r}\right)u\overline{v}\right]dx$$
$$+ \int_{\mathbb{R}^N} F\cdot\left(\frac{\overline{v}\nabla u}{r'} - \frac{u\nabla\overline{v}}{r}\right)dx. \tag{3.12}$$

Then it follows from (3.12) with $r := p$ and $v := \tilde{U}^{p-1}u|u|^{p-2} \in W^{1,1}(\mathbb{R}^N)$ that

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^N} (\tilde{A}u)\tilde{U}^{p-1}\overline{u}|u|^{p-2}\,dx \\
&= (p-1)(I_1+I_2) + \int_{\mathbb{R}^N} \tilde{U}^{p-1}|u|^{p-4}\langle a\,\mathrm{Im}\,(\overline{u}\nabla u), \mathrm{Im}\,(\overline{u}\nabla u)\rangle\,dx \\
&\quad + \int_{\mathbb{R}^N}\Big(\tilde{V} - \frac{\mathrm{div}F}{p}\Big)\tilde{U}^{p-1}|u|^p\,dx - \frac{p-1}{p}\int_{\mathbb{R}^N}\tilde{U}^{p-2}|u|^p(F\cdot\nabla)\tilde{U}\,dx, \qquad (3.13)
\end{aligned}$$

where we have set

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^N}\tilde{U}^{p-1}|u|^{p-4}\langle a\,\mathrm{Re}\,(\overline{u}\nabla u), \mathrm{Re}\,(\overline{u}\nabla u)\rangle\,dx, \\
I_2 &:= \int_{\mathbb{R}^N}\tilde{U}^{p-2}|u|^{p-2}\langle a\,\mathrm{Re}(\overline{u}\nabla u), \nabla\tilde{U}\rangle\,dx.
\end{aligned}$$

Here Young's inequality and (2.6′) apply to give

$$\begin{aligned}
I_1 - |I_2| &\ge I_1 - I_1^{1/2}\Big(\int_{\mathbb{R}^N}\tilde{U}^{p-3}\langle a\,\nabla\tilde{U}, \nabla\tilde{U}\rangle|u|^p\,dx\Big)^{1/2} \\
&\ge -\frac{1}{4}\int_{\mathbb{R}^N}\tilde{U}^{p-3}\langle a\,\nabla\tilde{U}, \nabla\tilde{U}\rangle|u|^p\,dx \\
&\ge -\frac{\lambda^2}{4}\|\tilde{U}u\|_{L^p}^p.
\end{aligned}$$

Now let $\eta \ge 0$. Then by virtue of (2.2′), (2.3′), (2.6′) and (2.7) we can rewrite (3.13) as

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^N}(\eta u + \tilde{A}u)\tilde{U}^{p-1}\overline{u}|u|^{p-2}\,dx \\
&\ge \int_{\mathbb{R}^N}\Big(\frac{\tilde{V}-\mathrm{div}F}{p} + \frac{\tilde{V}}{p'}\Big)\tilde{U}^{p-1}|u|^p\,dx \\
&\quad - \frac{p-1}{p}\beta\int_{\mathbb{R}^N}\tilde{U}^{p-2}\tilde{U}^{1/2}\langle a\nabla\tilde{U}, \nabla\tilde{U}\rangle^{1/2}|u|^p\,dx - (p-1)\frac{\lambda^2}{4}\|\tilde{U}u\|_{L^p}^p \\
&\ge \Big(\frac{\gamma_1}{p} + \frac{\gamma_\infty}{p'}\Big)\int_{\mathbb{R}^N}\tilde{U}\tilde{U}^{p-1}|u|^p\,dx \\
&\quad - \frac{p-1}{p}\beta\lambda\int_{\mathbb{R}^N}\tilde{U}^{p-3/2}\tilde{U}^{3/2}|u|^p\,dx - (p-1)\frac{\lambda^2}{4}\|\tilde{U}u\|_{L^p}^p.
\end{aligned}$$

Therefore we obtain

$$\mathrm{Re}\int_{\mathbb{R}^N}(\eta u + \tilde{A}u)\tilde{U}^{p-1}\overline{u}|u|^{p-2}\,dx \ge \Big(\frac{\gamma_1}{p} + \frac{\gamma_\infty}{p'} - \frac{p-1}{p}\beta\lambda - \frac{p-1}{4}\lambda^2\Big)\|\tilde{U}u\|_{L^p}^p.$$

Thus (3.10) is a consequence of Hölder's inequality.

Next we prove (3.11). It follows from (2.1′) and (2.4′) that

$$\|(F\cdot\nabla)u\|_{L^p}+\|\tilde{V}u\|_{L^p} \le \beta\,\|\tilde{U}^{1/2}\langle a\nabla u,\nabla u\rangle^{1/2}\|_{L^p}+c_0\|\tilde{U}u\|_{L^p}+c_1\|u\|_{L^p}. \quad (3.14)$$

Applying [10, Proposition 3.3] to our diffusion a and auxiliary function $\tilde{U} \ge C_\lambda > 0$, we see that for every $\varepsilon > 0$ there exists a constant $\tilde{C}_\varepsilon > 0$ depending only on N, p, ν and $\|a_{jk}\|_{W^{1,\infty}}$ such that

$$\beta\,\|\tilde{U}^{1/2}\langle a\nabla u,\nabla u\rangle^{1/2}\|_p \le \beta\,\varepsilon\|\mathrm{div}(a\nabla u)\|_{L^p} + \beta\,\tilde{C}_\varepsilon\|\tilde{U}u\|_{L^p}.$$

Plugging this inequality with $\varepsilon = (2\beta)^{-1}$ into (3.14), we have that

$$\begin{aligned}&\|(F\cdot\nabla)u\|_{L^p}+\|\tilde{V}u\|_{L^p}\\ &\le \frac{1}{2}\|(\eta+\tilde{A})u\|_{L^p}+\frac{1}{2}\Big(\|(F\cdot\nabla)u\|_{L^p}+\|\tilde{V}u\|_{L^p}\Big)\\ &\quad+(c_0+\beta\,\tilde{C}_{1/(2\beta)})\|\tilde{U}u\|_{L^p}+\Big(\frac{\eta}{2}+c_1\Big)\|u\|_{L^p},\quad \eta\ge 0. \qquad (3.15)\end{aligned}$$

Here it is worth noticing that since $A_{p,\min}$ is accretive in L^p, $\tilde{A}_{p,\min}$ is also accretive in L^p:

$$\eta\|u\|_{L^p} \le \|(\eta+\tilde{A})u\|_{L^p} \quad (\eta\ge 0). \qquad (3.16)$$

Therefore, (3.11) follows from (3.15) as a consequence of (3.10) and (3.16):

$$\begin{aligned}&\|(F\cdot\nabla)u\|_{L^p}+\|\tilde{V}u\|_{L^p}\\ &\le \|(\eta+\tilde{A})u\|_{L^p}+2(c_0+2\beta\,\tilde{C}_{1/(2\beta)})\|\tilde{U}u\|_{L^p}+(\eta+2c_1)\|u\|_{L^p}\\ &\le 2\left(1+\frac{c_0+\beta\,\tilde{C}_{1/(2\beta)}}{k_p(\lambda)}+\frac{c_1}{\eta}\right)\|(\eta+\tilde{A})u\|_{L^p},\quad \eta\ge 0.\end{aligned}$$

Finally, we prove (3.9). Condition (**H1**) and [6, Theorem 9.11] yield the well-known elliptic estimate: for every $u \in C_0^\infty(\mathbb{R}^N)$,

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} \le C(\|\mathrm{div}(a\nabla u)\|_{L^p}+\|u\|_{L^p}),$$

where C depends only on N, p, ν and $\|a_{jk}\|_{W^{1,\infty}}$. Now let $\eta \ge 1$. Then we can derive from (3.8) and (3.16) that

$$\begin{aligned}\|u\|_{W^{2,p}(\mathbb{R}^N)} &\le C(\|(\eta+\tilde{A})u\|_{L^p}+2\,\eta\|u\|_{L^p})+C(\|(F\cdot\nabla)u\|_{L^p}+\|\tilde{V}u\|_{L^p})\\ &\le C\Big(5+2\,\frac{c_0+\beta\,\tilde{C}_{1/(2\beta)}}{k_p(\lambda)}+\frac{2c_1}{\eta}\Big)\|(\eta+\tilde{A})u\|_{L^p},\quad \eta\ge 1.\end{aligned}$$

Thus we obtain (3.9). This completes the proof of Lemma 3. $\boxed{QED}$

Proof of Theorem 1. (**i**) Let $c_{q,\beta,\gamma}$ be the constant defined by (3.1). Then by [15, Theorem 1.3] we can conclude that for every $q \in (1,\infty)$, $A_{q,\max}$ is m-sectorial of type $S(c_{q,\beta,\gamma})$ in L^q, that is, $-A_{q,\max}$ generates an analytic contraction semigroup $\{e^{-zA_{q,\max}}\}$ on L^q on the closed sector $\overline{\Sigma}(\pi/2 - \tan^{-1} c_{q,\beta,\gamma})$. Moreover, we see from [15, Theorem 1.2] that $C_0^\infty(\mathbb{R}^N)$ is a core for $A_{p,\max}$. In fact, by condition (**H1**) it suffices to show that there exist a nonnegative auxiliary function $\Psi_q \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and a constant $\tilde{\beta} \geq 0$ such that

$$|\langle F(x), \xi\rangle| \leq \tilde{\beta}\Psi_q(x)^{1/2}\langle a(x)\xi,\xi\rangle^{1/2} \quad \text{a.a. } x \in \mathbb{R}^N, \xi \in \mathbb{C}^N, \tag{3.17}$$

$$V - \frac{\mathrm{div}F}{q} \geq \Psi_q \quad \text{a.e. on } \mathbb{R}^N. \tag{3.18}$$

Now set

$$\Psi_q(x) := \Big(\frac{\gamma_1}{q} + \frac{\gamma_\infty}{q'}\Big)U(x), \quad \tilde{\beta} := \beta\Big(\frac{\gamma_1}{q} + \frac{\gamma_\infty}{q'}\Big)^{-\frac{1}{2}}.$$

Then we see from conditions (2.1)–(2.3) with $\Omega = \mathbb{R}^N$ that (3.17) and (3.18) are satisfied:

$$\begin{aligned} |\langle F(x), \xi\rangle| &\leq \beta U(x)^{1/2}\langle a(x)\xi,\xi\rangle^{1/2} \\ &\leq \tilde{\beta}\Psi_q(x)^{\frac{1}{2}}\langle a(x)\xi,\xi\rangle^{1/2}, \\ \Psi_q(x) &\leq \frac{V(x) - \mathrm{div}F(x)}{q} + \frac{V(x)}{q'} \\ &= V(x) - \frac{\mathrm{div}F(x)}{q}, \end{aligned}$$

and hence we can apply [15, Theorem 1.3] to the triplet (a, F, V). The constant in (3.17) is reflected to that in (3.1). This completes the proof of assertion (**i**).

(**ii**) We want to construct a q-independent analytic continuation for $\{e^{-zA_{q,\max}}\}$. By virtue of Lemma 2 we can apply Lemma 1 (**i**) with $p_0 = 2$ to $\{e^{-zA_{2,\max}}\}$. Namely, the new family $\{T_q(t);\, t \geq 0\}$ of bounded linear operators on L^q defined as

$$(T_q(t)f)(x) = \int_{\mathbb{R}^N} k_t(x,y)f(y)\,dy, \quad f \in L^q(\mathbb{R}^N),\ \ t > 0,$$

with the kernel of $e^{-tA_{2,\max}}$ forms a C_0-semigroup on L^q for every $1 < q < \infty$. Denote by B_q the generator of $\{T_q(t)\}$ on L^q. Noting that $C_0^\infty(\mathbb{R}^N)$ is a core for $A_{q,\max}$, we deduce that $-B_q = A_{q,\max}$ and hence we obtain

$$T_q(t) = e^{-tA_{q,\max}} \quad \forall\, t \geq 0.$$

This implies by Theorem 1 (**i**) that $\{T_q(z)\} = \{e^{-zA_{q,\max}}\}$ is an analytic contraction semigroup on L^q on the closed sector $\overline{\Sigma}(\pi/2 - \tan^{-1} c_{q,\beta,\gamma})$.

Next let $q_0 \in (1, \infty)$ be as defined by

$$c_{q_0,\beta,\gamma} = \min_{1<q<\infty} c_{q,\beta,\gamma} = K_{\beta,\gamma}.$$

Then we see that $\{T_{q_0}(t)\}$ satisfies the assumption of Lemma 1 (**ii**) with

$$(p_0, \psi_0) := (q_0, \pi/2 - \tan^{-1} K_{\beta,\gamma}).$$

Therefore for every $p \in (1, \infty)$, $\{T_p(t)\}$ on L^p admits an analytic continuation to the sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$ such that

$$\|T_p(z)\|_{L^p} \le M_\varepsilon e^{\omega_0 \mathrm{Re} z}, \quad z \in \Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma} - \varepsilon), \tag{3.19}$$

where the constant M_ε depends only on ε, N, ν, β, γ_1, γ_∞ and $\|a_{jk}\|_{L^\infty}$. Consequently, the identity theorem for vector-valued analytic functions (see, e.g., [1, Theorem A.2]) implies that $\{T_p(z)\}$ is nothing but the analytic extension of $\{e^{-zA_{p,\max}}\}$ to the sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$ and hence using (3.19), we obtain (3.3). This completes the proof of assertion (**ii**).

(**iii**) It suffices to show that $A_{p,\max} = A_p$ if (**H3**) and (3.4) are satisfied with $\Omega = \mathbb{R}^N$. By definition we see that $A_p \subset A_{p,\max}$. Conversely, let $u \in D(A_{p,\max})$. Since $C_0^\infty(\mathbb{R}^N)$ is a core for $A_{p,\max}$, there exists a sequence $\{u_n\}$ in $C_0^\infty(\mathbb{R}^N)$ such that

$$u_n \to u, \quad Au_n \to A_{p,\max} u \quad \text{in } L^p \ (n \to \infty).$$

Applying Lemma 3 with $\xi = 1 + k_0 C_\lambda$, we see that for every $n \in \mathbb{N}$,

$$\|u_n\|_{W^{2,p}(\mathbb{R}^N)} + \|(F \cdot \nabla)u_n\|_{L^p} + \|Vu_n\|_{L^p} \le (C+1)\Big(5 + 2\frac{c_0 + \beta\, \tilde{C}_{1/(2\beta)}}{k_p}\Big)\|(\xi + A)u_n\|_{L^p}.$$

Letting $n \to \infty$, we see that $u \in W^{2,p}(\mathbb{R}^N) \cap D(F \cdot \nabla) \cap D(V) = D(A_p)$. This completes the proof of $A_p = A_{p,\max}$. $\boxed{QED}$

Example 2. We consider a typical one-dimensional Ornstein-Uhlenbeck operator

$$(A_\mu v)(x) := -v''(x) + xv'(x)$$

in L^p_μ (the L^p-space with respect to the invariant measure $e^{-x^2/2}dx$). Chill-Fašangová-Metafune-Pallara [4] show that the C_0-semigroup on L^p_μ generated by $-A_\mu$ is analytic in the sector $\Sigma(\tilde{\omega}_p)$ and that the angle $\tilde{\omega}_p = \pi/2 - \omega_p$ of analyticity is optimal.

Here, applying Theorem 1 (**ii**), we give another derivation of their angle ω_p. Using the isometry $u \mapsto e^{-x^2/2p}u$, we can transform A_μ into A:

$$(Au)(x) := -\frac{d^2u}{dx^2} + \Big(1 - \frac{2}{p}\Big)x\,\frac{d\,u}{dx} + \Big(\frac{p-1}{p^2}\,x^2 - \frac{1}{p}\Big)u$$

in the usual space $L^p(\mathbb{R}^N)$. Thus we obtain

$$a(x) \equiv 1, \quad F(x) := \Big(1 - \frac{2}{p}\Big)x, \quad V(x) := \frac{p-1}{p^2}\,x^2 - \frac{1}{p}$$

in our notation. Setting $U(x) := x^2$, the triplet $(a, F, V+1)$ satisfies conditions (**H1**) and (**H2**) with respective constants

$$\beta = |p-2|/p, \quad \gamma_1 = (p-1)/p^2 = \gamma_\infty.$$

In fact, (2.1)–(2.3) are computed as

$$\begin{aligned} |\langle F(x), \xi\rangle| &= p^{-1}|p-2|U(x)^{1/2}|\xi| \le \beta(U(x)+1)^{1/2}|\xi|, \\ (V(x)+1) - \mathrm{div}F(x) &= \frac{p-1}{p^2}U(x) + \frac{1}{p} \ge \gamma_1(U(x)+1), \\ V(x)+1 &= \frac{p-1}{p^2}U(x) + \frac{1}{p'} \ge \gamma_\infty(U(x)+1). \end{aligned}$$

This leads us to the angle ω_p introduced in Introduction:

$$K_{\beta,\gamma} = \inf_{1<q<\infty} \sqrt{\frac{(q-2)^2}{4(q-1)} + \frac{(p-2)^2}{4(p-1)}} = \frac{|p-2|}{2\sqrt{p-1}} = \tan\omega_p.$$

This shows that the domain of analyticity in this case is at least $\Sigma(\pi/2 - \omega_p)$ in a form of sector with vertex at the origin. Moreover, $U(x)$ satisfies (2.4) and (2.5) in (**H3**) with $c_0 = 1$ and $\lambda_0 = 0$, respectively. Hence A has a separation property (3.5).

4 The operators with local singularities

In this section we deal with the case $\Omega = \mathbb{R}^N \setminus \{0\}$. In this case $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ is not a core for $A_{p,\max}$ in general. In fact, $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ is not dense in $W^{2,p}(\mathbb{R}^N)$ if $p > N/2$. Therefore Theorem 1 (**i**) and (**ii**) may be false if $\mathbb{R}^N$ is replaced with $\mathbb{R}^N \setminus \{0\}$. Nevertheless we can show that Theorem 1 (**iii**) remains true even if $\Omega = \mathbb{R}^N \setminus \{0\}$ because $A_p = A_{p,\max}$ can be approximated by a family of operators $\{A_p^{(\delta)}\ ;\ \delta > 0\}$ with those properties in Theorem 1 (**i**), (**ii**) and (**iii**).

Theorem 2. *Let $1 < p < \infty$. Assume that conditions* (**H1**), (**H2**) *and* (**H3**) *are satisfied with $\Omega = \mathbb{R}^N \setminus \{0\}$. Let $K_{\beta,\gamma}$ be the constant determined by* (3.2)*. If* (3.4) *holds, then $\{e^{-zA_p}\}$ admits an analytic continuation to the sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$. In this case A_p has the separation property* (3.5).

Before proving Theorem 2, we introduce our approximation for the lower order coefficients. This is a modified version of Yosida approximation.

Lemma 4. *Let $\delta > 0$. Under the assumption in Theorem 2 put*

$$F_\delta(x) := \begin{cases} F(x)(1+\delta U(x))^{-2}, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{4.1}$$

$$U_\delta(x) := \begin{cases} U(x)(1+\delta U(x))^{-1}, & x \neq 0, \\ \delta^{-1}, & x = 0, \end{cases} \tag{4.2}$$

$$V_\delta(x) := \frac{V(x)}{1+\delta U(x)} + \frac{\gamma_1 \delta U(x)^2}{(1+\delta U(x))^2} + \frac{2\beta\lambda\delta(U(x)+C_\lambda)^2}{(1+\delta U(x))^3} \quad \text{a.a. } x \in \mathbb{R}^N, \tag{4.3}$$

where λ and C_λ are the constants in (2.6)*. Then*

$$F_\delta \in C^1(\mathbb{R}^N; \mathbb{R}^N), \quad U_\delta \in C^1(\mathbb{R}^N; \mathbb{R}^N), \quad V_\delta \in L^\infty(\mathbb{R}^N; \mathbb{R}) \tag{4.4}$$

and the triplet (a, F_δ, V_δ) and U_δ satisfy

$$F_\delta \to F \ \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^N), \qquad V_\delta \to V \ \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}; \mathbb{R}) \tag{4.5}$$

and (2.1)–(2.3) *with $\Omega = \mathbb{R}^N$:*

$$|\langle F_\delta(x), \xi\rangle| \le \beta U_\delta(x)^{1/2} \langle a(x)\xi, \xi\rangle^{1/2}, \quad x \in \mathbb{R}^N,\ \xi \in \mathbb{C}^N, \tag{4.6}$$

$$V_\delta(x) - \operatorname{div} F_\delta(x) \ge \gamma_1 U_\delta(x) \quad \text{a.a. } x \in \mathbb{R}^N, \tag{4.7}$$

$$V_\delta(x) \ge \gamma_\infty U_\delta(x) \quad \text{a.a. } x \in \mathbb{R}^N. \tag{4.8}$$

Moreover, for $\delta \le 1/C_\lambda$, one has (2.4) *and* (2.6) *for the triplet (a, F_δ, V_δ) :*

$$V_\delta(x) \le (c_0 + \gamma_1 + 2\beta\lambda) U_\delta(x) + c_1 + 2\beta\lambda C_\lambda, \tag{4.9}$$

$$\langle a(x)\nabla U_\delta(x), \nabla U_\delta(x)\rangle^{1/2} \le \lambda (U_\delta(x) + C_\lambda)^{3/2}. \tag{4.10}$$

Proof. We can verify (4.4) and (4.5) by a simple computation. Now we prove conditions **(H2)** and **(H3)** for the approximated triplet (a, F_δ, V_δ). Since the original triplet (a, F, V) satisfies conditions (2.1) and (2.3) with $\Omega = \mathbb{R}^N \setminus \{0\}$, we see that (4.6) and (4.8) are satisfied: the case of $x = 0$ is clear and

$$|\langle F_\delta(x), \xi\rangle| = \frac{|\langle F(x), \xi\rangle|}{(1+\delta U(x))^2} \le \frac{\beta U(x)^{1/2}\langle a(x)\xi, \xi\rangle^{1/2}}{(1+\delta U(x))^{1/2}} = \beta U_\delta(x)^{1/2} \langle a(x)\xi, \xi\rangle^{1/2},$$

$$V_\delta(x) \ge \frac{V(x)}{1+\delta U(x)} \ge \frac{\gamma_\infty U(x)}{1+\delta U(x)} = \gamma_\infty U_\delta(x).$$

Furthermore, combining (2.2) and (2.7), we obtain (4.7):

$$\begin{aligned}
&V_\delta(x) - \mathrm{div} F_\delta(x) \\
&\geq \frac{V(x) - \mathrm{div} F(x)}{(1+\delta U(x))^2} + \gamma_1 \frac{\delta U(x)^2}{(1+\delta U(x))^2} + 2\delta \frac{\beta\lambda \tilde{U}(x)^2 - |(F\cdot\nabla)\tilde{U}(x)|}{(1+\delta U(x))^3} \\
&\geq \gamma_1 \frac{U(x)}{(1+\delta U(x))^2} + \gamma_1 \frac{\delta U(x)^2}{(1+\delta U(x))^2} \\
&= \gamma_1 U_\delta(x).
\end{aligned}$$

Now we prove (4.9) and (4.10). We see from (2.4) that for every $\delta \in (0, 1/C_\lambda]$,

$$\begin{aligned}
V_\delta(x) &\leq (c_0 + \gamma_1) U_\delta(x) + c_1 + 2\beta\lambda \Big(\frac{\delta C_\lambda + \delta U(x)}{1 + \delta U(x)} \Big) \frac{U(x) + C_\lambda}{(1+\delta U(x))^2} \\
&\leq (c_0 + \gamma_1 + 2\beta\lambda) U_\delta(x) + c_1 + 2\beta\lambda C_\lambda.
\end{aligned}$$

It follows from the estimate (2.6) for the original triplet (a, F, V) that

$$\begin{aligned}
\langle a(x)\nabla U_\delta(x), \nabla U_\delta(x)\rangle^{1/2} &= \frac{\langle a(x)\nabla U(x), \nabla U(x)\rangle^{1/2}}{(1+\delta U(x))^2} \\
&\leq \frac{\lambda}{(1+\delta U(x))^{1/2}} \Big(\frac{U(x) + C_\lambda}{1 + \delta U(x)} \Big)^{3/2} \\
&\leq \lambda (U_\delta(x) + C_\lambda)^{3/2}.
\end{aligned}$$

This completes the proof of Lemma 4. $\boxed{QED}$

Proof of Theorem 2. In view of (3.4) we fix $\lambda > \lambda_0$ satisfying

$$(p-1)\lambda \Big(\frac{\beta}{p} + \frac{\lambda}{4} \Big) < \frac{\gamma_1}{p} + \frac{\gamma_\infty}{p'}.$$

For $\delta > 0$ let F_δ, V_δ and U_δ be as (4.1)–(4.3). Then Lemma 4 implies that the approximate triplet (a, F_δ, V_δ) satisfies **(H2)** and **(H3)** with $\Omega = \mathbb{R}^N$ and (3.4). Thus the triplet (a, F_δ, V_δ) satisfies the assumption in Theorem 1 (**iii**). Therefore we can define a family $\{A_p^{(\delta)} \,;\, \delta > 0\}$ approximate to A_p in L^p:

$$\begin{cases}
D(A_p^{(\delta)}) := W^{2,p}(\mathbb{R}^N), \\
A_p^{(\delta)} u := -\mathrm{div}(a\nabla u) + (F_\delta \cdot \nabla) u + V_\delta u, \quad u \in D(A_p^{(\delta)}).
\end{cases}$$

Let ω_0 be the constant as in Theorem 1 (**ii**) depending only on N, $\|a_{jk}\|_{L^\infty}$ and $\|\nabla a_{jk}\|_{L^\infty}$. Then $-A_p^{(\delta)}$ generates a bounded analytic semigroup $\{e^{-z(\omega_0 + A_p^{(\delta)})}\}$ in the open sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$, with two norm bounds:

$$\|e^{-zA_p^{(\delta)}}\|_{L^p} \leq 1, \quad z \in \overline{\Sigma}(\pi/2 - \tan^{-1} c_{p,\beta,\gamma}),$$

and for every $\varepsilon > 0$ there exists a constant $M_\varepsilon \geq 1$ such that

$$\|e^{-zA_p^{(\delta)}}\|_{L^p} \leq M_\varepsilon e^{\omega_0 \mathrm{Re} z}, \quad z \in \Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma} - \varepsilon), \tag{4.11}$$

where M_ε depends only on ε, N, ν, β, γ_1, γ_∞ and $\|a_{jk}\|_{L^\infty}$. Moreover, $A_p^{(\delta)}$ has the separation property (3.5): for every $u \in W^{2,p}(\mathbb{R}^N)$ $(= D(A_p^{(\delta)}))$,

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} + \|(F_\delta \cdot \nabla)u\|_{L^p} + \|U_\delta u\|_{L^p} \leq C\|u + A_p^{(\delta)}u\|_{L^p}, \tag{4.12}$$

where C is independent of $\delta \in (0, 1/C_\lambda]$.

Next we prove the m-sectoriality of A_p. Let $v \in D(A_p)$. Then by the definition of $A_p^{(\delta)}$ we have $v \in D(A_p^{(\delta)})$ and $A_p^{(\delta)}v \to A_p v$ $(\delta \downarrow 0)$ in L^p. We see from the sectoriality of $A_p^{(\delta)}$ that A_p is also sectorial in L^p. It remains to prove the maximality: $R(I + A_p) = L^p$. Let $f \in L^p$. We see from the m-accretivity of $A_p^{(\delta)}$ that for every $\delta > 0$ there exists $u_\delta \in D(A_p^{(\delta)})$ such that

$$u_\delta - \mathrm{div}(a\nabla u_\delta) + (F_\delta \cdot \nabla)u_\delta + V_\delta u_\delta = f.$$

Hence (4.12) yields that for every $\delta \in (0, 1/C_\lambda]$,

$$\|u_\delta\|_{W^{2,p}(\mathbb{R}^N)} + \|(F_\delta \cdot \nabla)u_\delta\|_{L^p} + \|U_\delta u_\delta\|_{L^p} \leq C\|f\|_{L^p}. \tag{4.13}$$

It follows from (4.13) that there exist a subsequence $\{u_{\delta_m}\}_m$ with $\delta_m \downarrow 0$ $(m \to \infty)$ and a function $u \in W^{2,p}(\mathbb{R}^N) \cap D(U)$ such that

$$\begin{aligned} u_{\delta_m} &\to u \quad (m \to \infty) \quad \text{weakly in } W^{2,p}(\mathbb{R}^N), \\ U_{\delta_m} u_{\delta_m} &\to Uu \ (m \to \infty) \quad \text{weakly in } L^p(\mathbb{R}^N). \end{aligned}$$

It follows from (2.4) that $Vu \in L^p$. The Rellich-Kondrachov theorem implies that

$$u_{\delta_m} \to u \quad \text{in } W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N).$$

Using Fatou's lemma, we see that

$$\|(F \cdot \nabla)u\|_{L^p}^p \leq \liminf_{m\to\infty} \|(F_{\delta_m} \cdot \nabla)u_{\delta_m}\|_{L^p}^p \leq C^p\|f\|_{L^p}^p.$$

Thus we have $u \in D(A_p)$. By (4.5) in Lemma 4 we deduce that

$$\begin{aligned} (F_{\delta_m} \cdot \nabla)u_{\delta_m} &\to (F \cdot \nabla)u \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}), \\ V_{\delta_m} u_{\delta_m} &\to Vu \qquad\qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}) \end{aligned}$$

and hence we obtain $u + A_p u = f$, that is, $R(I + A_p) = L^p$. This completes the proof of the m-sectoriality of A_p.

Consequently, the Hille-Yosida generation theorem modified by Goldstein [7, Theorem 1.5.9] implies that $-A_p$ generates an analytic contraction semigroup $\{e^{-tA_p}\}$ on L^p. Furthermore, applying Trotter's convergence theorem (see, e.g., [5, Theorem III.4.8]), we deduce that for every $f \in L^p$ and $t \geq 0$,

$$e^{-tA_p^{(\delta)}} f \to e^{-tA_p} f \ \text{ in } L^p.$$

Finally, by Vitali's theorem (see, e.g., [1, Theorem A.5]) we see from (4.11) that $\{e^{-tA_p}\}$ admits an analytic continuation to the sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$. Moreover,

$$\|e^{-zA_p}\|_{L^p} \leq 1, \quad z \in \overline{\Sigma}(\pi/2 - \tan^{-1} c_{p,\beta,\gamma}),$$

and for every $\varepsilon > 0$,

$$\|e^{-zA_p}\|_{L^p} \leq M_\varepsilon e^{\omega_0 \mathrm{Re} z}, \quad z \in \Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma} - \varepsilon). \tag{4.14}$$

Noting that (4.14) implies the continuity at the origin, we finish the proof. $\boxed{QED}$

Example 3 (A case where $\gamma_1 \neq \gamma_\infty$)**.** We consider the following operator

$$Au = -\Delta u + \frac{bx}{|x|^2} \cdot \nabla u + \frac{c}{|x|^2},$$

that is, (a, F, V) and Ω in our notation are given by

$$a_{jk}(x) := \delta_{jk}, \quad F(x) := \frac{bx}{|x|^2}, \quad V(x) := \frac{c}{|x|^2}, \quad \Omega = \mathbb{R}^N \setminus \{0\};$$

note that this operator has a singularity at the origin. Taking the auxiliary function U as $U(x) := |x|^{-2}$, we can see that the respective constants in (**H2**) are given by

$$\beta = |b|, \quad \gamma_1 = c - b(N-2), \quad \gamma_\infty = c.$$

Thus $\gamma_1 \neq \gamma_\infty$ if $N \neq 2$ and $b \neq 0$. We also have $\lambda_0 = 2$ (see Example 1). Hence if b, c and p satisfy (3.4), that is, if

$$p - 1 + \frac{2}{p}|b| = (p-1)\lambda_0\left(\frac{\beta}{p} + \frac{\lambda_0}{4}\right) < \frac{\gamma_1}{p} + \frac{\gamma_\infty}{p'} = c - \frac{b(N-2)}{p}$$

holds, then we can apply Theorem 2 to the operator A and hence the conclusion of Remark 3 yields that $c_{2,\beta,\gamma} > K_{\beta,\gamma}$.

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander: Vector-Valued Laplace Transform and Cauchy Problems, Monographs in Mathematics **96**, Birkhauser Verlag, Basel, 2001.

[2] W. Arendt, A. F. M. ter Elst: *Gaussian estimates for second order elliptic operators with boundary conditions*, J. Operator Theory **38** (1997), 87–130.

[3] W. Arendt, G. Metafune, D. Pallara: *Schrödinger operators with unbounded drift*, J. Operator Theory **55** (2006), 185–211.

[4] R. Chill, E. Fašangová, G. Metafune, D. Pallara: *The sector of analyticity of the Ornstein-Uhlenbeck semigroup on L^p spaces with invariant measure*, J. London Math. Soc. (2) **71** (2005), 703–722.

[5] K.J. Engel, R. Nagel: One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics **194**, Springer-Verlag, New York, 2000.

[6] D. Gilbarg, N. S. Trudinger: Elliptic Partial Differential Equations of Second Order, Second Edition, Springer-Verlag, Berlin, 1983.

[7] J.A. Goldstein: Semigroups of Linear Operators and Applications, Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 1985.

[8] M. Hieber: *Gaussian estimates and holomorphy of semigroups on L^p spaces*, J. London Math. Soc. 2nd Ser. **54** (1996), 148–160.

[9] Y. Maeda, N. Okazawa: *Holomorphic families of Schrödinger operators in L^p*, SUT J. Math. **47** (2011), 185–216.

[10] G. Metafune, D. Pallara, J. Prüss, R. Schnaubelt: *L^p-theory for elliptic operators on $\mathbb{R}^d$ with singular coefficients*, Z. Anal. Anwendungen **24** (2005), 497–521.

[11] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt: *L^p-regularity for elliptic operators with unbounded coefficients*, Adv. Differential Equations **10** (2005), 1131–1164.

[12] N. Okazawa: *Sectorialness of second order elliptic operators in divergence form*, Proc. Amer. Math. Soc. **113** (1991), 701–706.

[13] E. M. Ouhabaz: *Gaussian estimates and holomorphy of semigroups*, Proc. Amer. Math. Soc. **123** (1995), 1465–1474.

[14] E. M. Ouhabaz: Analysis of Heat Equations on Domains, London Math. Soc. Monogr. Ser. **31**, Princeton University Press, Princeton, NJ, 2005.

[15] M. Sobajima: *L^p-theory for second order elliptic operators with unbounded coefficients*, J. Evol. Equ. **12** (2012), 957–971.